WESLEY COLLEGE
By daring & by doing

YEAR 12 MATHEMATICS METHODS
Logarithmic functions and
calculus involving logarithmic functions
Test 4

Name: ______________________ Marks: /50

Calculator Free (32 marks)

1. [5 marks]

Determine the exact values of *a*, *b* and *c*.

a) $3^a = 11$

$$a = \frac{\log 11}{\log 3} = \frac{\ln 11}{\ln 3} = \log_3 11$$

[1]

b) $\log_2(2b-1) = 4$

$$2^4 = 16 = 2b - 1$$
$$2b = 17$$
$$b = 8.5$$

[2]

c) $\log_2 4^{2c} = \log_3 27^{c+1}$

$$2c\log_2 4 = (c+1)\log_3 27$$
$$4c = 3c + 3$$
$$c = 3$$

[2]

2. [3 marks]

Express $\log 9 - 3\log\sqrt{3} + \log 81$ in the form $k\log 3$ where k is an exact fraction.

$$\begin{aligned} &\log 9 - 3\log\sqrt{3} + \log 81 \\ &= \log 3^2 - 3\log 3^{\frac{1}{2}} + \log 3^4 \\ &= 2\log 3 - \frac{3}{2}\log 3 + 4\log 3 = \frac{9}{2}\log 3 \end{aligned}$$

3. [7 marks]

Differentiate:

a) $5\log_e x$ $\quad \frac{d}{dx}(5\ lnx) = \frac{5}{x}$

[1]

b) $\log_e\left(2x^3 - 5x\right)$ $\quad \frac{d}{dx}\ln\left(2x^3 - 5x\right) = \frac{6x^2 - 5}{2x^3 - 5x}$

[1]

c) $\ln\left[(2x-1)(4x+3)\right]$ (<u>no need to simplify</u> your answer)

$$\frac{d}{dx}\ln\left[(2x-1)(4x+3)\right] = \frac{\frac{d}{dx}\left(8x^2 + 2x - 3\right)}{\left[(2x-1)(4x+3)\right]} = \frac{16x+2}{\left[(2x-1)(4x+3)\right]}$$

or

$$\frac{d}{dx}\ln\left[(2x-1)(4x+3)\right] = \frac{d}{dx}\ln(2x-1) + \ln(4x+3) = \frac{2}{2x-1} + \frac{4}{4x+3}$$

[2]

d) $\frac{\ln x}{x^2}$ (<u>simplify</u> your answer)

$$\frac{d}{dx}\left(\frac{\ln x}{x^2}\right) = \frac{x^2 \times \frac{1}{x} - 2x \times \ln x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$$

[3]

4. [4 marks]

Show that $\int_2^4 \left(x - 4x^{-1} + 4x^{-3}\right)\,dx = \dfrac{51}{8} - 4\ln 4$

$$\int_2^4 \left(x - 4x^{-1} + 4x^{-3}\right)\,dx = \left[\frac{x^2}{2} - 4\ln x - 2x^{-2}\right]_2^4$$

$$= \left[8 - 4\ln 4 - \frac{1}{8}\right] - \left[2 - 4\ln 2 - \frac{1}{2}\right]$$

$$= 6 - 8\ln 2 + 4\ln 2 + \frac{3}{8}$$

$$= \frac{51}{8} - 4\ln 2$$

5. [5 marks]

Show the only solution to $3^{2x+1} - 11\left(3^x\right) - 4 = 0$ is $x = \log_3 4$.

Let $y = 3^x$

$$3y^2 - 11y - 4 = 0 \qquad \left(3y+1\right)\left(y-4\right) = 0$$

$$y = -\frac{1}{3} \text{ or } y = 4$$

$$3^x \neq -\frac{1}{3} \quad 3^x = 4 \quad \Rightarrow \quad x\log 3 = \log 4$$

$$x = \frac{\log 4}{\log 3} = \log_3 4$$

6. [8 marks]

a) Given that

$$p = \log_2 x \text{ and } q = \log_2 y$$

find expressions in terms of p and q for

i. $\log_2 (x^2 y)$

$$= \log_2 x^2 + \log_2 y$$

$$= 2\log_2 x + \log_2 y$$

$$= 2p + q$$

[2]

ii. $\log_2 \left(\frac{\sqrt{y}}{x^3} \right)$

$$= \frac{1}{2}\log_2 y - 3\log_2 x$$

$$= \frac{1}{2}q - 3p$$

[2]

b) Hence, or otherwise, solve the simultaneous equations

$$\log_2 (x^2 y) = 2$$

$$\log_2 \left(\frac{\sqrt{y}}{x^3} \right) = -11$$

$2p + q = 2$ $\quad$ $2p + q = 2$

$\frac{1}{2}q - 3p = -11 \ (\times 2)$ $\quad$ $-6p + q = -22$

$8p = 24$ $\quad$ $p = 3,\ q = -4$

$-4 = \log_2 y \quad \Rightarrow y = 2^{-4} = \frac{1}{16}$

$3 = \log_2 x \quad \Rightarrow x = 2^3 = 8$

[4]

End of Part A

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Calculator Section (18 marks)

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7. [2 marks]

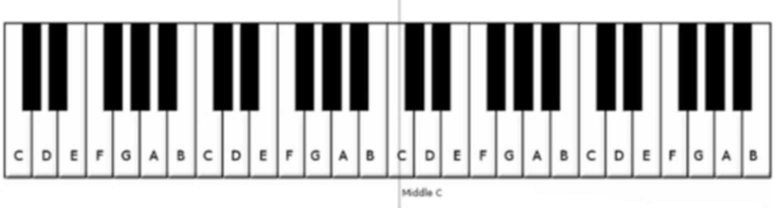
Biologists use the logarithmic model $n = k\ \log(A)$ to estimate the number of species, n, that live in a region of area A km^2. In the model, k represents a constant.
If 2800 species live in a rainforest of 500 square kilometres, then how many species will be left when half of this rainforest is destroyed by logging?

$$2800 = k\ \log(500) \Rightarrow k = \frac{2800}{\log(500)}$$

$$n = \frac{2800}{\log(500)} \times \log(250) = 2487.70161 \approx 2500\ (\textit{nearest}\ 100)$$

[2]

8. [2 marks]



In the modern scale of musical notes the note names repeat every octave, each note is **double** the frequency of the note of the same name in the octave below. The A note below middle C has a standard frequency of 440 Hertz.

There are actually 12 different notes, including flats and sharps, in an octave. This is called a chromatic scale and the ratio of the frequency of one note to the previous note in the chromatic scale is a constant.

a) What is the ratio of a musical note to the previous note in the chromatic scale?

$\sqrt[12]{2} = 1.059463094$

[1]

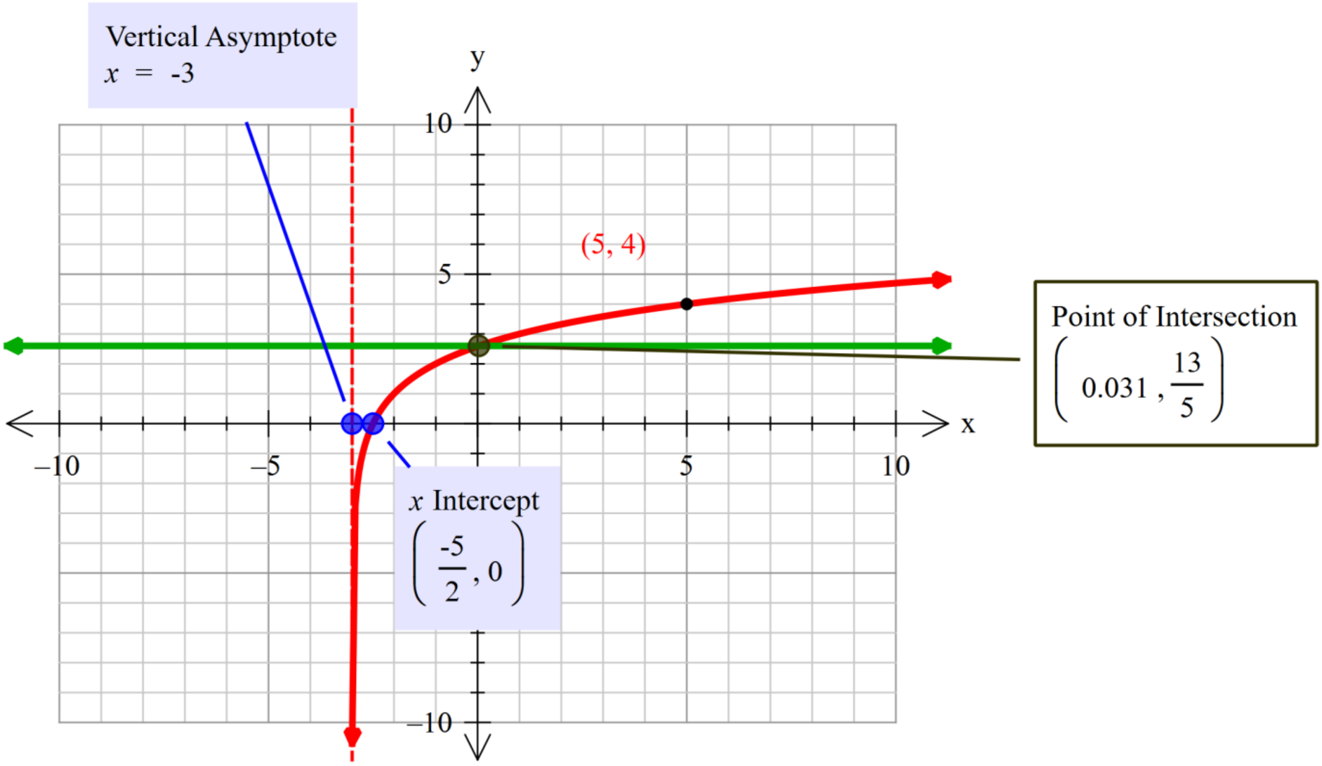
b) What is the frequency of middle C?

$$440 \times \sqrt[4]{2} = 523.2511306 \text{ Hz}$$

[1]

9. [5 marks]

a) Sketch the graph of $y = \log_2(x+3)+1$ on the grid below, labelling important features.



[3]

b) Show how you use your graph to solve $\log_2(x+3) = 1.6$

$\log_2(x+3) = 1.6$

$\log_2(x+3)+1 = 2.6$ x-coordinate of point of intersection = 0.031

Draw $y = 2.6$ *on graph*

[2]

10. [8 marks]

A small colony of black peppered moths lives on a small isolated island. In summer the population begins to increase. If t is the number of days after 12 midnight on 1 January, the equation that best models the number of moths in the colony at any given time is

$$N(t) = 500\ \ln(21t + 3) \qquad 0 \le t \le 40$$

a) What is the population of the species on 1 January?

$N(0) = 500\ \ln(3) = 549$

[1]

b) What is the population of moths after 30 days?

$N(30) = 500\ \ln(21 \times 30 + 3) \approx 3225$

[1]

c) On which day is the population first greater than 2000?

$500\ \ln(21t + 3) > 2000$

t >2.457 i.e. 3 January

[1]

A related species, the white peppered moth, shares the same habitat. It reproduces in a similar pattern to the black peppered moth, with its population modelled by

$$P(t) = P \ln(Qt + 3) \qquad 0 \le t \le 40$$

d) The initial population is 769 and the population when $t = 15$ is 2750. Find the value of P, correct to the nearest whole number and Q, correct to 3 decimal places.

$$\left.\begin{array}{l} 769 = P\ln(3) \\ 2750 = P\ln(15Q + 3) \end{array}\right\}_{P,Q}$$

$P = 699.97, \qquad Q = 3.18944$

$P = 700 \qquad Q = 3.189$

[2]

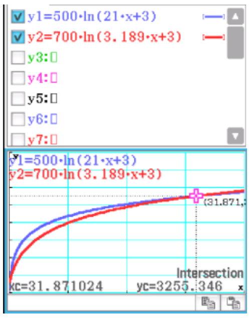
Please turn over …

e) The populations of P(t) and N(t) are briefly equal on the first day. Determine an approximate time when the populations of the black and white peppered moths will again be the same.

Evidence of graph sketch or solving the equation

$500\ln(21t+3)=700\ln(3.189t+3)$

when $t\approx 31.87$ *i.e.* 1 *February*

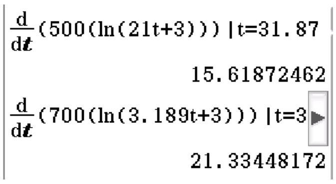


☑ y1=500·ln(21·x+3)
☑ y2=700·ln(3.189·x+3)
☐ y3:▯
☐ y4:▯
☐ y5:▯
☐ y6:▯
☐ y7:▯

[2]

f) What are the population growth rates at this time?

16 black moths/day
21 white moths/day



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d/dt(500(ln(21t+3)))|t=31.87
                    15.61872462
d/dt(700(ln(3.189t+3)))|t=3
                    21.33448172
```

[2]

End of Part B